Unbiased estimators

Recap: Cramer-Rao lower bound

Let $X_1, \dots, X_n$ be a sample from a distribution with probability function $f(x|\theta)$, and let $\hat{\theta}$ be an unbiased estimator of $\theta \in \mathbb{R}$. Then, under regularity conditions,

$$Var(\hat{\theta}) \geq \frac{1}{\mathcal{I}(\theta)}$$

Cramér-Rao lower bound (CRLB)

Example

Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

> Calculate the Cramer-Rao lower bound for the variance of an unbiased estimator of σ^2. Does the sample variance $s^2 = \frac{1}{n-1}\sum_i (X_i - \overline{X})^2$ attain the Cramer-Rao lower bound?

Want $\mathcal{I}(\sigma^2)$

$$\ell(\mu,\sigma^2) = \log\left(\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2\right\} \right)$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$Var(u(\sigma^2)) = Var\left(\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2\right) \quad \left(\frac{1}{2\sigma^4}\right)^2 \sum_{i=1}^{n} Var\left((x_i-\mu)^2\right)$$

or

$$\frac{\partial^2 \ell}{\partial \sigma^4} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$-\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \sigma^4}\right] = -\left(\frac{n}{2\sigma^4} - \frac{n}{\sigma^4}\right) = \frac{n}{2\sigma^4} = \mathcal{I}(\sigma^2)$$

$$\text{CRLB} = \frac{2\sigma^4}{n}$$

$$Var(s^2) = \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n}$$

Attaining the CRLB

Let $X_1, \dots, X_n$ be iid with probability function $f(x|\theta)$, and suppose the regularity conditions for the CRLB hold. Let $\hat{\theta}$ be an unbiased estimator of θ. Then $\hat{\theta}$ attains the CRLB if and only if

$$u(\theta) = a(\theta)[\hat{\theta} - \theta]$$

(for some function $a(\theta)$).

Pf: Proof of the CRLB uses the fact that

$$[\mathrm{Cov}(\hat{\theta}, u(\theta))]^2 \leq \mathrm{Var}(\hat{\theta})\,\mathrm{Var}(u(\theta))$$

The CRLB is attained when this inequality is an equality, i.e.

$$[\mathrm{Cov}(\hat{\theta}, u(\theta))]^2 = \mathrm{Var}(\hat{\theta})\,\mathrm{Var}(u(\theta))$$

Lemma: $(\mathrm{Cov}(X,Y))^2 = \mathrm{Var}(X)\,\mathrm{Var}(Y)$ (i.e. $|\mathrm{Corr}(X,Y)| = 1$) if and only if $X - \mathbb{E}[X] = c(Y - \mathbb{E}[Y])$

So, $[\mathrm{Cov}(\hat{\theta}, u(\theta))]^2 = \mathrm{Var}(\hat{\theta})\,\mathrm{Var}(u(\theta))$ if and only if $u(\theta) - \mathbb{E}[u(\theta)] = a(\theta)[\hat{\theta} - \mathbb{E}[\hat{\theta}]] \iff u(\theta) = a(\theta)[\hat{\theta} - \theta]$

Example: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$U(\sigma^2) = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (X_i - \mu)^2$$

$$= \frac{n}{2\sigma^4} \left(\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{n} - \sigma^2 \right)$$

If $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$ then $\mathbb{E}[\hat{\sigma}^2] = \sigma^2$

and $a(\sigma^2) = \frac{n}{2\sigma^4}$ then $U(\sigma^2) = a(\sigma^2)(\hat{\sigma}^2 - \sigma^2)$

$\Rightarrow$ $\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$ attains the CRLB <u>if</u> μ is known

If μ is <u>unknown</u>, then CRLB cannot be attained

Sufficient statistics

> Given an unbiased estimator, can I improve its variance?

<u>Def</u>: Let $X_1, \dots, X_n$ be a sample from a distribution $f(x \mid \theta)$. Let $T \equiv T(X_1, \dots, X_n)$ be a statistic. If the conditional distribution of $X_1, \dots, X_n \mid T$ does not depend on θ, then T is a <u>sufficient statistic</u> for θ

<u>Ex</u>: Suppose $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. Let $T = \sum_{i=1}^n X_i$

$T \sim \text{Poisson}(n\lambda)$

$$f(X_1, \dots, X_n \mid T, \lambda) = \frac{f(X_1, \dots, X_n, T \mid \lambda)}{f(T \mid \lambda)} = \frac{f(X_1, \dots, X_n \mid \lambda)}{f(T \mid \lambda)}$$

$$= \frac{\cancel{e^{-n\lambda}} \, \cancel{\lambda^{\sum_i X_i}} \,/\, \prod_i X_i!}{\cancel{e^{-n\lambda}} \, (n\lambda)^T \,/\, T!} = \frac{T! \, n^T}{\prod_i X_i!}$$

(does not depend on λ)

$\Rightarrow T$ is sufficient

Rao-Blackwell

Let θ be a parameter of interest, and $\tau(\theta)$ some function of θ. Let $\hat{\tau}$ be some unbiased estimator of $\tau(\theta)$, and T a sufficient statistic for θ.

Let $\tau^* = \mathbb{E}[\hat{\tau} \mid T]$. Then:

1. $\mathbb{E}[\tau^*] = \tau$ (unbiased)
2. $\mathrm{Var}(\tau^*) \leq \mathrm{Var}(\hat{\tau})$

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$.