Method of moments estimators

Course so far

- Maximum likelihood estimation
- Logistic regression
- Asymptotics
- Asymptotic properties of MLEs
- Hypothesis testing
- Confidence intervals

> **Common theme:** Likelihoods and MLEs
>
> **Question:** Why maximum likelihood estimation?

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. How could I estimate θ?

① $\hat{\theta}_{MLE} = X_{(n)}$

② median $U[0,\theta] = \frac{\theta}{2}$

$\hat{\theta}$ = sample median $\times 2$

③ $\mathbb{E}[X] = \frac{\theta}{2} \Rightarrow \hat{\theta} = 2\bar{X}$

④ $\mathbb{E}[X^2] = \frac{\theta^2}{12} + \frac{\theta^2}{4} = \frac{\theta^2}{3} \Rightarrow \hat{\theta} = \sqrt{\frac{3}{n}\sum_i X_i^2}$

⑤ $\hat{\theta} = 5$ (probably a terrible estimate)

$$\Rightarrow \quad \hat{a} = \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)}$$
$$\hat{b} = \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)}$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[a, b]$. How could I estimate a and b?

① Fix a, estimate b, alternate

② MLE: $\hat{a}_{MLE} = X_{(1)}$ $\quad \hat{b}_{MLE} = X_{(n)}$

③ $\mathbb{E}[X] = \frac{a+b}{2} = \mu_1 \qquad \hat{\mu}_1 = \frac{1}{n}\sum_i X_i = \bar{X}$

$\mathbb{E}[X^2] = \frac{1}{3}(a^2 + ab + b^2) = \mu_2 \qquad \hat{\mu}_2 = \frac{1}{n}\sum_i X_i^2$

$$b = 2\mu_1 - a$$
$$\Rightarrow \mu_2 = \frac{1}{3}\left(a^2 + a(2\mu_1 - a) + (2\mu_1 - a)^2\right)$$
$$= \frac{1}{3}\left(a^2 - 2a\mu_1 + 4\mu_1^2\right)$$
$$\Rightarrow 3\mu_2 - 3\mu_1^2 = (a - \mu_1)^2$$
$$\Rightarrow a = \mu_1 - \sqrt{3(\mu_2 - \mu_1^2)} \qquad b = \mu_1 + \sqrt{3(\mu_2 - \mu_1^2)}$$

Method of moments

Let $X_1, \ldots, X_n$ be a sample from a distribution with probability function $f(x|\theta_1, \ldots, \theta_k)$, with k parameters $\theta_1, \ldots, \theta_k$.

Let

$$\mu_1 = \mathbb{E}[X] = g_1(\theta_1, \ldots, \theta_k) \qquad \hat{\mu}_1 = \frac{1}{n}\sum_i X_i$$

$$\mu_2 = \mathbb{E}[X^2] = g_2(\theta_1, \ldots, \theta_k) \qquad \hat{\mu}_2 = \frac{1}{n}\sum_i X_i^2$$

$$\vdots$$

$$\mu_k = \mathbb{E}[X^k] = g_k(\theta_1, \ldots, \theta_k) \qquad \hat{\mu}_k = \frac{1}{n}\sum_i X_i^k$$

The method of moments (MoM) approach estimates $\theta_1, \ldots, \theta_k$ by the solutions to

$$\hat{\mu}_1 = g_1(\hat{\theta}_1, \ldots, \hat{\theta}_k)$$

$$\hat{\mu}_2 = g_2(\hat{\theta}_1, \ldots, \hat{\theta}_k)$$

$$\vdots$$

$$\hat{\mu}_k = g_k(\hat{\theta}_1, \ldots, \hat{\theta}_k)$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

Find the method of moments estimates $\widehat{\mu}$ and $\widehat{\sigma}^2$.

$$\mu_1 = \mu \qquad \hat{\mu}_1 = \bar{X} \qquad \hat{\mu} = \bar{X}$$

$$\mu_2 = \mathrm{Var}(X) + (\mathbb{E}[X])^2 \qquad \hat{\mu}_2 = \frac{1}{n}\sum_i X_i^2$$

$$= \sigma^2 + \mu^2$$

$$\sigma^2 = \mu_2 - \mu_1^2 \quad \Rightarrow \quad \hat{\sigma}^2 = \hat{\mu}_2 - \hat{\mu}_1^2$$

$$= \frac{1}{n}\sum_i X_i^2 - (\bar{X})^2$$

$$= \frac{1}{n}\sum_i (X_i - \bar{X})^2$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Gamma(\alpha, \beta)$, i.e.
$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$. Then

$$\mu_1 = \mathbb{E}[X] = \frac{\alpha}{\beta} \qquad \mu_2 = \mathbb{E}[X^2] = \left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta^2}$$

> Use the method of moments to estimate α and β.

$$\mu_1 = \frac{\alpha}{\beta} \qquad \mu_2 = \left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta^2}$$

$$\alpha = \beta\mu_1$$

$$\Rightarrow \mu_2 = \left(\frac{\beta\mu_1}{\beta}\right)^2 + \frac{\beta\mu_1}{\beta^2}$$

$$= \mu_1^2 + \frac{\mu_1}{\beta}$$

$$\beta\mu_2 = \beta\mu_1^2 + \mu_1$$

$$\beta(\mu_2 - \mu_1^2) = \mu_1$$

$$\beta = \frac{\mu_1}{\mu_2 - \mu_1^2} \qquad \Rightarrow \qquad \alpha = \frac{\mu_1^2}{\mu_2 - \mu_1^2}$$

$$\Rightarrow \quad \hat{\alpha} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2} \qquad \hat{\beta} = \frac{\hat{\mu}_1}{\hat{\mu}_2 - \hat{\mu}_1^2} \qquad \hat{\alpha}, \hat{\beta} > 0 \quad (\hat{\mu}_2 > \hat{\mu}_1^2)$$