Confidence intervals

Warm-up: Pivots

Let $X_1, \dots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

Find a pivotal quantity $\overbrace{Q(X_1, \dots, X_n, \theta)}^{\text{distribution does not depend on } \theta}$ and construct a $1 - \alpha$ confidence interval for θ using the pivotal quantity.

Hints:

- Begin with the maximum likelihood estimate of θ, which is $\hat{\theta} = \dfrac{n}{\sum_{i=1}^{n} X_i}$
- If $X \sim Exponential(\theta)$, then $cX \sim Exponential\left(\frac{\theta}{c}\right)$
- $Exponential\left(\frac{1}{2}\right) = \chi^2_2$

$X_1, \dots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$

if $X \sim \text{Exp}(\theta)$

$\Rightarrow cX \sim \text{Exp}\left(\frac{\theta}{c}\right)$

$Q(X_1, \dots, X_n, \theta) = 2\theta \sum_{i=1}^{n} X_i$

$2\theta X \sim \text{Exp}\left(\frac{\theta}{2\theta}\right) = \text{Exp}\left(\frac{1}{2}\right)$

$X \sim \text{Exponential}(\theta) \Rightarrow 2\theta X \sim \chi_2^2$

$= \chi_2^2$

$\Rightarrow 2\theta \sum_{i=1}^{n} X_i \sim \chi_{2n}^2 = \text{Gamma}\left(n, \frac{1}{2}\right)$

Next, want a, b st $P_\theta\left(a \le 2\theta \sum_{i=1}^{n} X_i \le b\right) = 1 - \alpha$

e.g. $a = 0$ $\quad b = \chi_{2n,\alpha}^2$

$a = \chi_{2n,1-\frac{\alpha}{2}}^2$, $b = \chi_{2n,\frac{\alpha}{2}}^2$

$\Rightarrow a \le 2\theta \sum_{i=1}^{n} X_i \le b$

$\Rightarrow \left[\frac{a}{2\sum_{i=1}^{n} X_i}, \frac{b}{2\sum_{i=1}^{n} X_i}\right]$

By continuous mapping, if $\hat{\theta} \approx N(\theta, var(\hat{\theta}))$

Wald CI

$\Rightarrow g(\hat{\theta}) \xrightarrow{d} \ldots$

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with density $f(x|\theta) = \theta e^{-\theta x}$.

MLE: $\frac{1}{\bar{x}} = \frac{n}{\sum_{i=1}^{n} x_i}$

Wald CI: by asymptotic normality of MLE,

$\hat{\theta} \approx N(\theta, \mathcal{I}^{-1}(\theta))$ $\quad \mathcal{I}(\theta) = \frac{n}{\theta^2}$

$\Rightarrow \hat{\theta} \approx N(\theta, \frac{\theta^2}{n})$ $\quad \Rightarrow \mathcal{I}^{-1}(\theta) = \frac{\theta^2}{n}$

variance depends on θ

Can we find a transformation $g(\hat{\theta})$ such that $Var(g(\hat{\theta}))$ does not depend on θ

$\hat{\theta} \to \theta \Rightarrow g(\hat{\theta}) \to g(\theta)$

Delta method

Suppose $\hat{\theta}$ is an estimate of $\theta \in \mathbb{R}$, such that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2)$$

for some σ^2, and g is a continuously differentiable function with $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2)$$

Proof sketch:

- First-order Taylor expansion of $g(\hat{\theta})$ around θ
- Slutsky's theorem

Pf:

1) Taylor expansion:

$$g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta}-\theta)$$

$$\Rightarrow \quad g(\hat{\theta}) - g(\theta) \approx g'(\theta)(\hat{\theta}-\theta)$$

$$\Rightarrow \quad \sqrt{n}\left(g(\hat{\theta})-g(\theta)\right) \approx \underbrace{g'(\theta)\underbrace{\sqrt{n}(\hat{\theta}-\theta)}_{\xrightarrow{d} N(0,\sigma^2)}}$$

(Slutsky's)

$$\xrightarrow{d} g'(\theta)N(0,\sigma^2)$$

$$= N(0,[g'(\theta)]^2\sigma^2)$$

//

Variance stabilizing transformations

Example

Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} Bernoulli(p)$.