Confidence intervals

Recap: confidence sets

Let $\theta \in \Theta$ be a parameter of interest, and $X_1, \ldots, X_n$ a sample. A set $C(X_1, \ldots, X_n) \subseteq \Theta$ is a $1 - \alpha$ **confidence set** for θ if

$$\inf_{\theta \in \Theta} P_\theta(\theta \in C(X_1, \ldots, X_n)) = 1 - \alpha$$

Last time: create a confidence set by inverting a test

$$C(X_1, \ldots, X_n) = \{ \theta_0 : (X_1, \ldots, X_n) \notin R(\theta_0) \}$$

$\uparrow$
rejection region for
α-level test of
$H_0 : \theta = \theta_0$ vs.
$H_A : \theta \neq \theta_0$

Using confidence sets to test hypotheses

Theorem: Let $\theta \in \Theta$ and let $C(X_1, \dots, X_n)$ a $1-\alpha$ confidence set.

For any $\theta_0 \in \Theta$, let

$$R(\theta_0) = \{(X_1, \dots, X_n) : \theta_0 \notin C(X_1, \dots, X_n)\}$$

The test which rejects $H_0 : \theta = \theta_0$ when $(X_1, \dots, X_n) \in R(\theta_0)$ is an α-level test

Pf: $P_{\theta_0}((X_1, \dots, X_n) \in R(\theta_0)) = P_{\theta_0}(\theta_0 \notin C(X_1, \dots, X_n))$

$$= 1 - \underbrace{P_{\theta_0}(\theta_0 \in C(X_1, \dots, X_n))}_{\geq 1-\alpha}$$

$$\leq 1 - (1-\alpha) = \alpha$$

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Example: Inverting the t-test

Suppose that $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. We want to construct a $1 - \alpha$ confidence interval for μ.

> Construct a $1 - \alpha$ confidence interval for μ by inverting the t-test.

reject $H_0: \mu = \mu_0$ when $\left| \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \right| > t_{n-1, \frac{\alpha}{2}}$

$$\Rightarrow 1-\alpha \text{ CI} = \left\{ \mu_0 : -t_{n-1,\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \leq t_{n-1,\frac{\alpha}{2}} \right\}$$

$$= \left[\bar{X} - t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}},\ \bar{X} + t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right]$$

Another way to view this: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\underbrace{Q(X_1, \dots, X_n, \mu)}_{\text{pivotal quantity}} = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Distribution of $Q(X_1, \dots, X_n, \mu)$ does not depend on μ

To construct a $1-\alpha$ confidence set for μ:

Find a, b st $P_\mu(a \le Q(X_1, \dots, X_n, \mu) \le b) = 1 - \alpha$

do not depend on μ

$\Rightarrow$ $1-\alpha$ confidence set for μ is $\{\mu : a \le Q(X_1, \dots, X_n, \mu) \le b\}$

e.g. for $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$, $a = -t_{n-1, \frac{\alpha}{2}}$ $b = t_{n-1, \frac{\alpha}{2}}$

Pivotal quantities

Let $X_1, \ldots, X_n$ be a sample and θ be an unknown parameter. A function $Q(X_1, \ldots, X_n, \theta)$ is a called a pivot if the distribution of $Q(X_1, \ldots, X_n, \theta)$ does not depend on θ

Find a, b st $P_\theta(a \leq Q(X_1, \ldots, X_n, \theta) \leq b) = 1-\alpha$

Then a $1-\alpha$ confidence set for θ is

$$\{\theta : a \leq Q(X_1, \ldots, X_n, \theta) \leq b\}$$

Example

Let $X_1, \dots, X_n \overset{iid}{\sim}$ Uniform $[0, \theta]$

Want a $1-\alpha$ confidence set for θ.

$\hat{\theta} = X_{(n)}$, so maybe we can use $X_{(n)}$ to create a confidence set

$$P(X_{(n)} \leq t) = \left(P(X_i \leq t)\right)^n = \left(\frac{t}{\theta}\right)^n \quad (t \in [0, \theta])$$

$$\underbrace{Q(X_1, \dots, X_n, \theta) = \frac{X_{(n)}}{\theta}}_{\text{pivot!}} \Rightarrow P\left(\frac{X_{(n)}}{\theta} \leq t\right) = \left(\frac{t \cdot \theta}{\theta}\right)^n = t^n$$

$\Rightarrow$ choose a, b st $P\left(a \leq \frac{X_{(n)}}{\theta} \leq b\right) = 1-\alpha$

$$\Rightarrow \frac{1}{b} \leq \frac{\theta}{X_{(n)}} \leq \frac{1}{a} \Rightarrow \frac{X_{(n)}}{b} \leq \theta \leq \frac{X_{(n)}}{a}$$

$\Rightarrow$ Interval $= \left[\frac{X_{(n)}}{b}, \frac{X_{(n)}}{a}\right]$ ($1-\alpha$ CI)

e.g. $b = 1$, $a = \alpha^{\frac{1}{n}}$ $\Rightarrow$ interval $= \left[X_{(n)}, \frac{X_{(n)}}{\alpha^{\frac{1}{n}}}\right]$

(equivalent to inverting LRT)

a, b st $P\left(a \le \frac{X_{(n)}}{\theta} \le b\right) = 1-\alpha$

$\frac{X_{(n)}}{\theta} \in [0,1] \Rightarrow a, b \in [0,1]$

$P\left(\frac{X_{(n)}}{\theta} \le b\right) = b^n$ $\qquad P\left(\frac{X_{(n)}}{\theta} \le a\right) = a^n$

$\Rightarrow$ find a, b st $b^n - a^n = 1-\alpha$

E.g. $b = 1 \Rightarrow 1 - a^n = 1-\alpha$

$\Rightarrow a = \alpha^{\frac{1}{n}}$