Interval estimation

Motivation

Suppose we have data $(X_1, Y_1), \dots, (X_n, Y_n)$ with

$$Y_i \sim Bernoulli(p_i)$$

$$\log\left(\frac{p_i}{1 - p_i}\right) = \beta^T X_i$$

So far, we have discussed:

- Finding point estimates $\widehat{\beta}$
- Testing hypotheses about the true (but unknown) parameters β

> What are the limitations of point estimates and hypothesis tests for inference about β?

Confidence interval

```
...
##                  Estimate  Std. Error  z value  Pr(>|z|)
## (Intercept)   2.6415063  0.1213233    21.77   <2e-16 ***
## WBC          -0.2892904  0.0134349   -21.53   <2e-16 ***
## PLT          -0.0065615  0.0005932   -11.06   <2e-16 ***
## ---
...
```

> How would I calculate a 95% confidence interval for β_1 (the change in the log odds of dengue for a one-unit increase in WBC, holding PLT fixed)?

$\hat{\beta}_1 \pm z_{\frac{\alpha}{2}} SE(\hat{\beta}_1)$ ← $1-\alpha$ Wald CI

95% CI : $-0.289 \pm 1.96\,(0.0134)$

$= (-0.315, -0.262)$

β_1 is either in (-0.315, -0.262) or not
(no "95% probability" for a specific interval)

Confidence interval

```
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## ---
...
```

coverage probability

95% confidence interval for β_1: (-0.315, -0.262)

> How do I interpret this confidence interval?

95% confident: if we take many samples and we calculate many intervals, 95% should contain the true (unknown) parameter

"we are 95% confident that a one-unit increase in WBC is associated w/ a decrease in log odds of dengue by between 0.262 and 0.315"

$$\hat{\theta} \sim N(\theta, Var(\hat{\theta})) \Rightarrow \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \sim N(0,1)$$

Deriving the coverage probability

$(1-\alpha)$ Wald interval: $\hat{\theta} \pm z_{\frac{\alpha}{2}} SE(\hat{\theta})$

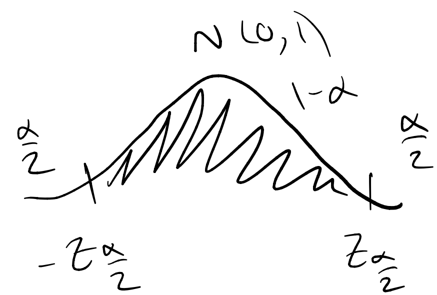
$$\hookrightarrow P\left(\hat{\theta} - z_{\frac{\alpha}{2}} SE(\hat{\theta}) \le \theta \le \hat{\theta} + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right) = 1 - \alpha$$

endpoints are random (function of data)

$$P\left(\hat{\theta} - z_{\frac{\alpha}{2}} SE(\hat{\theta}) \le \theta \le \hat{\theta} + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right)$$

$$= P\left(-z_{\frac{\alpha}{2}} SE(\hat{\theta}) \le \hat{\theta} - \theta \le z_{\frac{\alpha}{2}} SE(\hat{\theta})\right)$$

$$= P\left(-z_{\frac{\alpha}{2}} \le \underbrace{\frac{\hat{\theta} - \theta}{SE(\hat{\theta})}}_{\sim N(0,1)} \le z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$



Note: $\theta_0 \in \left[\hat{\theta} - z_{\frac{\alpha}{2}} SE(\hat{\theta}),\ \hat{\theta} + z_{\frac{\alpha}{2}} SE(\hat{\theta})\right]$

$$\Leftrightarrow \left| \frac{\hat{\theta} - \theta_0}{SE(\hat{\theta})} \right| \le z_{\frac{\alpha}{2}}$$

i.e. the α-level Wald test of $H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$ fails to reject

Formal definition

Let $\theta \in \Theta$ be a parameter of interest, and $X_1, \dots, X_n$ a sample. Let $C(X_1, \dots, X_n) \subseteq \Theta$ be a set constructed from $X_1, \dots, X_n$ ($\Rightarrow$ $C(X_1, \dots, X_n)$ is a random set)

$C(X_1, \dots, X_n)$ is a <u>$1-\alpha$ confidence set</u> for θ if

$$\inf_{\theta \in \Theta} P_\theta\left(\theta \in C(X_1, \dots, X_n)\right) = 1-\alpha$$

(i.e. $\forall \theta \in \Theta$, $P_\theta(\theta \in C(X_1, \dots, X_n)) \geq 1-\alpha$)

α-level test of $H_0: \theta = \theta_0$: $P_{\theta_0}((X_1, \dots, X_n) \in R(\theta_0)) \le \alpha$

rejection region

Inverting a test

<u>Theorem</u>: Let $\theta \in \Theta$ be a parameter of interest.

For each value of $\theta_0 \in \Theta$, consider testing $H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$, and let $R(\theta_0)$ be the rejection region for a level α test.

Let $C(X_1, \dots, X_n) = \{\theta_0 : (X_1, \dots, X_n) \notin R(\theta_0)\}$

Then $C(X_1, \dots, X_n)$ is a $1-\alpha$ confidence set for θ.

<u>Pf</u>: $\theta \in C(X_1, \dots, X_n) \iff (X_1, \dots, X_n) \notin R(\theta)$

$$\Rightarrow P_\theta(\theta \in C(X_1, \dots, X_n)) = P_\theta((X_1, \dots, X_n) \notin R(\theta))$$

$$= 1 - P_\theta((X_1, \dots, X_n) \in R(\theta))$$

$$\geq 1 - \alpha \qquad (\alpha\text{-level test}) \quad //$$

Example

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} Uniform[0, \theta]$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$$

Find the LRT statistic for this test.

Reject H_0 when $\dfrac{\sup_{\theta} L(\theta \mid X)}{L(\theta_0 \mid X)} > K$

$L(\theta \mid X) = \left(\frac{1}{\theta}\right)^n \mathbb{1}\{X_{(n)} \leq \theta\}$ $\qquad \hat{\theta}_{MLE} = X_{(n)}$

$\Rightarrow$ reject H_0 when $\dfrac{\theta_0^n}{X_{(n)}^n \mathbb{1}\{\theta_0 \geq X_{(n)}\}} > K$

$\Rightarrow$ reject H_0 when $\theta_0 < X_{(n)}$ or when $\dfrac{\theta_0}{X_{(n)}} > K^{\frac{1}{n}}$

reject H_0 when $\theta_0 < X_{(n)}$ or $\dfrac{\theta_0}{X_{(n)}} > K^{\frac{1}{n}}$

$\Rightarrow$ fail to reject H_0 when $X_{(n)} \leq \theta_0 \leq X_{(n)} K^{\frac{1}{n}}$

confidence set $= [X_{(n)}, X_n K^{\frac{1}{n}}] = [X_{(n)}, X_{(n)} K']$

So need K' st $P_\theta(\theta \in [X_{(n)}, X_{(n)} K']) \geq 1-\alpha$

$$
\begin{aligned}
P_\theta(\theta \in [X_{(n)}, X_{(n)} K']) &= P_\theta(X_{(n)} K' \geq \theta) \\
&= 1 - P_\theta(X_{(n)} K' < \theta) \\
&= 1 - P_\theta\left(X_{(n)} < \frac{\theta}{K'}\right) \\
&= 1 - \left(\frac{\theta / K'}{\theta}\right)^n = 1 - \left(\frac{1}{K'}\right)^n
\end{aligned}
$$

$\Rightarrow$ choose K' st $\left(\frac{1}{K'}\right)^n = \alpha \Rightarrow K' = \dfrac{1}{\alpha^{\frac{1}{n}}}$

$1-\alpha$ CI for θ: $\left[X_{(n)}, \dfrac{X_{(n)}}{\alpha^{1/n}}\right]$