Neyman-Pearson and likelihood ratio tests

Recap: Neyman-Pearson test

Let $X_1, \ldots, X_n$ be a sample from a distribution with probability function f, and parameter θ. To test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

the Neyman-Pearson test rejects H_0 when

$$\frac{L(\theta_1 | \mathbf{X})}{L(\theta_0 | \mathbf{X})} > k,$$

where k is chosen so that $\beta(\theta_0) = \alpha$.

Recap: Neyman-Pearson lemma

The Neyman-Pearson test is a *uniformly most power* level α test of $H_0 : \theta = \theta_0$ vs. $H_A : \theta = \theta_1$.

Proof: N-P rejects when $\frac{f(x|\theta_1)}{f(x|\theta_0)} > k$

Let β_{NP} denote power of N-P test. Choose k st $\beta_{NP}(\theta_0) = \alpha$

Let β^* denote power of another level α test $(\Rightarrow \beta^*(\theta_0) \leq \alpha)$

WTS $\beta_{NP}(\theta_1) \geq \beta^*(\theta_1)$

Let ϕ_{NP} denote N-P rejection function : $\phi_{NP}(x) = \begin{cases} 1 & \text{N-P reject} \\ 0 & \text{N-P fails to reject} \end{cases}$

Let ϕ^* rejection function for the other α-level test

$$\phi^*(x) = \begin{cases} 1 & \text{if other test reject} \\ 0 & \text{if other test fails to reject} \end{cases}$$

$$\beta_{NP}(\theta_0) = P_{\theta_0}(\text{reject } H_0) = P_{\theta_0}(\phi_{NP}(x) = 1)$$

$$= \int_X \phi_{NP}(x) f(x|\theta_0)\, dx$$

$$\beta^*(\theta_0) = \int_X \phi^*(x)\, f(x|\theta_0)\, dx$$

(N-P rejects when $f(x|\theta_1) > k f(x|\theta_0)$
$\iff f(x|\theta_1) - k f(x|\theta_0) > 0$)

we know $\beta_{NP}(\theta_0) = \alpha$, $\beta^*(\theta_0) \le \alpha$, so $\beta_{NP}(\theta_0) - \beta^*(\theta_0) \ge 0$

$$\Rightarrow \int_X (\phi_{NP}(x) - \phi^*(x))\, f(x|\theta_0)\, dx \ge 0$$

Now let's look at $\int_X \underbrace{(\phi_{NP}(x) - \phi^*(x))}\underbrace{(f(x|\theta_1) - k f(x|\theta_0))}\, dx$

$= 0$ if tests agree | > 0 if $\phi_{NP}(x) = 1$

$= 1$ N-P rejects, other test fails to reject | ≤ 0 if $\phi_{NP}(x) = 0$

$= -1$ N-P fails to reject, other test rejects

so $(\phi_{NP}(x) - \phi^*(x))(f(x|\theta_1) - k f(x|\theta_0))$
$$\begin{cases} = 0 & \text{if tests agree} \\ > 0 & \text{if N-P rejects \& other test fails to reject} \\ \ge 0 & \text{if N-P fails to reject, \& other test rejects} \end{cases}$$

$$\int_X (\phi_{NP}(x) - \phi^*(x))(f(x|\theta_1) - k f(x|\theta_0))\, dx \ge 0$$

$$\Rightarrow \int_X (\phi_{NP}(x) - \phi^*(x))\, f(x|\theta_1)\, dx \ge k \int_X (\phi_{NP}(x) - \phi^*(x))\, f(x|\theta_0)\, dx$$

$$\Rightarrow \beta_{NP}(\theta_1) - \beta^*(\theta_1) \ge k(\beta_{NP}(\theta_0) - \beta^*(\theta_0)) \ge 0$$

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Composite hypotheses with a UMP test

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu > \mu_0$$

Claim: the Wald test is a uniformly most powerful level α test for these hypotheses.

Pf: The Wald test rejects when $\bar{X}_n > \mu_0 + \frac{\sigma z_\alpha}{\sqrt{n}}$

$\Rightarrow \beta_{wald}(\mu_0) = \alpha \Rightarrow$ wald test is a level α test

Let $\mu_1 > \mu_0$, let β^* be power of another level α test

WTS $\beta_{wald}(\mu_1) \geq \beta^*(\mu_1)$

we showed last class that $\bar{x}_n > \mu_0 + \frac{\sigma z_\alpha}{\sqrt{n}}$

$\Leftrightarrow \frac{f(x|\mu_1)}{f(x|\mu_0)} > k$

$\Rightarrow$ for each μ_1, the wald test is equivalent to N-P test

$\Rightarrow \beta_{wald}(\mu_1) = \beta_{NP}(\mu_1) \geq \beta^*(\mu_1)$

For most scenarios, there is not a UMP test

Composite hypotheses without a UMP test

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu \neq \mu_0$$

Claim: there is no uniformly most powerful level α test for these hypotheses.

Rough sketch (see Casella & Berger for more details)

If $\mu_1 > \mu_0$, N-P test rejects when $\bar{X} > \mu_0 + \frac{\sigma z_\alpha}{\sqrt{n}}$

Let β_1 be the power of this test $\Rightarrow \beta_1(\mu_1) \geq \beta^*(\mu_1)$ for all other α-level test

If $\mu_2 < \mu_0$, N-P rejects when $\bar{X} < \mu_0 - \frac{\sigma z_\alpha}{\sqrt{n}}$

Let β_2 be the associated power $\Rightarrow \beta_2(\mu_2) \geq \beta^*(\mu_2)$

If test 1 is UMP, $\beta_1(\mu) \geq \beta^*(\mu) \quad \forall \mu \neq \mu_0$

If test 2 is UMP, $\beta_2(\mu) \geq \beta^*(\mu) \quad \forall \mu \neq \mu_0$

we will show $\beta_1(\mu_1) > \beta_2(\mu_1)$, and $\beta_2(\mu_2) > \beta_1(\mu_2)$

Now,

$$\beta_1(\mu_1) = P_{\mu_1}\left(\bar{X} > \frac{\sigma z_\alpha}{\sqrt{n}} + \mu_0\right)$$

$$= P_{\mu_1}\left(\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} > z_\alpha + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right)$$

$$> P(Z > z_\alpha) \qquad Z \sim N(0,1) \qquad (\mu_0 - \mu_1 < 0)$$

$$= P(Z < -z_\alpha)$$

$$> P\left(Z < -z_\alpha + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right) \qquad (\mu_0 - \mu_1 < 0)$$

$$= P_{\mu_1}\left(\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} < -z_\alpha + \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}}\right)$$

$$= P_{\mu_1}\left(\bar{X} < \frac{-\sigma z_\alpha}{\sqrt{n}} + \mu_0\right)$$

$$= \beta_2(\mu_1)$$

$\Rightarrow \quad \beta_1(\mu_1) > \beta_2(\mu_1)$

Similarly, $\beta_1(\mu_2) < \beta_2(\mu_2)$

So our candidates for the UMP test are <u>not</u> UMP

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The likelihood ratio test

Let $X_1, \dots, X_n$ be a sample from a distribution with parameter $\theta \in \mathbb{R}^d$. We wish to test $H_0: \theta \in \Theta_0$ vs. $H_A: \theta \in \Theta_1$.

The likelihood ratio test (LRT) rejects H_0 when

$$\frac{\sup_{\theta \in \Theta_1} L(\theta \mid X)}{\sup_{\theta \in \Theta_0} L(\theta \mid X)} > k,$$

where k is chosen so that $\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$. We wish to test $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda \neq \lambda_0$.