Neyman-Pearson lemma

Wald test for normal mean

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

Wald test: reject H_0 if

$$\frac{\bar{X}_n - \mu_0}{\sigma / \sqrt{n}} > z_\alpha$$

i.e. $\bar{X}_n > \mu_0 + \frac{\sigma z_\alpha}{\sqrt{n}} = c_0$

Wald test for normal mean

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects if

$$\overline{X}_n > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

We know that $\beta(\mu_0) = \alpha$ for this test.

> Does there exist a different test, with power function $\beta^*(\mu)$, such that $\beta^*(\mu_0) \leq \alpha$ and $\beta^*(\mu_1) > \beta(\mu_1)$?

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i^2-2\mu x_i+\mu^2) = \sum_{i=1}^{n} x_i^2 - 2\mu n\bar{X} + n\mu^2$$

Rearranging the Wald test for a population mean

rejects H_0 when $\bar{X}_n > c_0$

$$\iff 2n\bar{X}(\mu_0-\mu_1) < 2nc_0(\mu_0-\mu_1) \qquad (\mu_0-\mu_1<0)$$

$$\iff 2n\bar{X}(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2 < 2nc_0(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i^2 + 2n\bar{X}(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2 < 2nc_0(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff \sum_{i=1}^{n}(x_i-\mu_1)^2 - \sum_{i=1}^{n}(x_i-\mu_0)^2 < 2nc_0(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2$$

$$\iff -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_1)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2 > -\frac{(2nc_0(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2)}{2\sigma^2}$$

$$\iff \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_i(x_i-\mu_1)^2\right\}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_i(x_i-\mu_0)^2\right\}} > \underbrace{\exp\left\{-\frac{(2nc_0(\mu_0-\mu_1) - n\mu_0^2 + n\mu_1^2)}{2\sigma^2}\right\}}_{=k_0}$$

$$\iff \frac{f(x_1,\dots,x_n\mid\mu_1)}{f(x_1,\dots,x_n\mid\mu_0)} > k_0$$

Rearranging the Wald test for a population mean

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects if $\overline{X}_n > \overset{c_0}{\cancel{k_0}}$, which is equivalent to rejecting when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} = \frac{f(X_1, \ldots, X_n|\mu_1)}{f(X_1, \ldots, X_n|\mu_0)} > k_0$$

Intuition: Reject H_0 if the likelihood of μ_1 is sufficiently greater than the likelihood of μ_0.

Neyman-Pearson test

Let $X_1, \dots, X_n$ be a sample from a distribution with probability function f, and parameter θ.

To test $H_0: \theta = \theta_0$ vs. $H_A: \theta = \theta_1$, the Neyman-Pearson test rejects H_0 when

$$\frac{L(\theta_1 \mid X)}{L(\theta_0 \mid X)} = \frac{f(X \mid \theta_1)}{f(X \mid \theta_0)} > k,$$

where k is chosen so that $\beta(\theta_0) = \alpha$.

Neyman-Pearson lemma

Lemma: The Neyman-Pearson test is a uniformly most powerful level α test of $H_0: \theta = \theta_0$ vs. $H_A: \theta = \theta_1$

(i.e., $\beta_{NP}(\theta_1) \geq \beta^*(\theta_1)$ for any other α-level test).

Def: Consider testing $H_0: \theta \in \Theta_0$ vs. $H_A: \theta \in \Theta_1$

Let C_α be the set of level-α tests for these hypotheses.

A test in C_α is the uniformly most powerful level α test

if $\beta(\theta) \geq \beta^*(\theta) \quad \forall \, \theta \in \Theta_1$, for all other tests

β^* = power of another test in C_α

Example

Let $\mathbf{X} = X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with σ^2 known. We wish to test

$$H_0 : \mu = \mu_0 \qquad H_A : \mu = \mu_1$$

where $\mu_1 > \mu_0$.

The Wald test rejects when

$$\frac{L(\mu_1|\mathbf{X})}{L(\mu_0|\mathbf{X})} > k,$$

where k is chosen such that $\beta(\mu_0) = \alpha$.

$\Rightarrow$ Wald test for these hypotheses is a uniformly most powerful test

Example

Let $X_1, \ldots, X_n \overset{iid}{\sim} Exponential(\theta)$, with pdf $f(x|\theta) = \theta e^{-\theta x}$. We want to test

$$H_0 : \theta = \theta_0 \qquad H_A : \theta = \theta_1,$$

where $\theta_1 < \theta_0$. The Neyman-Pearson test rejects when

$$\frac{L(\theta_1|\mathbf{X})}{L(\theta_0|\mathbf{X})} > k.$$

> Find k such that the test has size α.

$$\beta(\theta_0) = \alpha \Rightarrow P_{\theta_0}\left(\frac{L(\theta_1|x)}{L(\theta_0|x)} > k \right) = \alpha$$

$$\frac{L(\theta_1|x)}{L(\theta_0|x)} = \frac{\theta_1^n e^{-\theta_1 \sum_i x_i}}{\theta_0^n e^{-\theta_0 \sum_i x_i}} > k$$

$$\Rightarrow \exp\left\{ \sum_i x_i (\theta_0 - \theta_1) \right\} > k \left(\frac{\theta_0}{\theta_1} \right)^n$$

$$\Rightarrow (\theta_0 - \theta_1) \sum_i x_i > \log k + n \log\left(\frac{\theta_0}{\theta_1} \right)$$

$$\Rightarrow \sum_i x_i > \frac{\log k + n \log\left(\frac{\theta_0}{\theta_1} \right)}{\theta_0 - \theta_1}$$

we know $\sum_{i=1}^{n} x_i \sim \text{Gamma}(n, \theta)$ $\left(f_{\sum x_i}(x) = \frac{\theta^n}{\Gamma(n)} x^{n-1} e^{-\theta x} \right)$

Under H_0, $\sum_i x_i \sim \text{Gamma}(n, \theta_0)$

$\Rightarrow$ choose k such that $\frac{\log k + n \log\left(\frac{\theta_0}{\theta_1} \right)}{\theta_0 - \theta_1}$ = upper α quantile of Gamma(n, θ_0)