t-tests

Exam 2

- released Monday, March 13

  Due Monday, March 20

- Covers HW 4 – 6 (up through Wald tests)

  I will return feedback on HW 5, 6 over spring break

Next steps

(includes proportions)

> So far, we have discussed the Wald test in detail. What other hypothesis tests have you seen in statistics courses?

- t- tests
- F- tests
- χ^2 tests
- lots of others...

Different test statistics w/ different distributions

Recap: power function

$$H_0 : \theta \in \Theta_0 \qquad H_A : \theta \in \Theta_1$$

Suppose we reject H_0 when $(X_1, \ldots, X_n) \in R$. The **power function** $\beta(\theta)$ is

$$\beta(\theta) = P_\theta((X_1, \ldots, X_n) \in R)$$

Example: $X_1, \ldots, X_n$ iid from a population with mean μ and variance σ^2.

$$H_0 : \mu = \mu_0 \qquad H_A : \mu > \mu_0$$

$$\beta(\mu) \approx 1 - \Phi\left(z_\alpha - \frac{(\mu - \mu_0)}{\sigma/\sqrt{n}}\right)$$

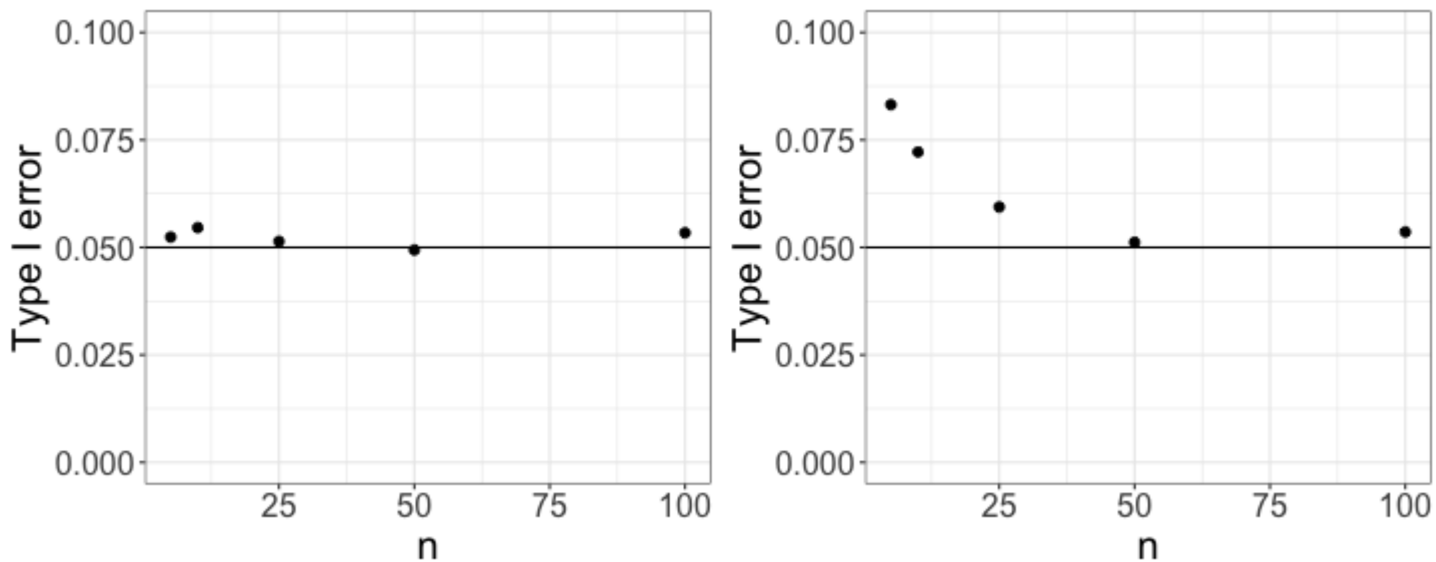
Class activity, Part I

https://sta711-s23.github.io/class_activities/ca_lecture_21.html

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

Class activity

$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ $\frac{\sqrt{n}(\bar{X}-\mu)}{S}$



If we reject $H_0 : \mu = \mu_0$ when $\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s} > z_\alpha$, why does type I error increase as n decreases?

$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \xrightarrow{d} N(0,1)$ as $n \to \infty$. But for small n, $\frac{\sqrt{n}(\bar{X}-\mu)}{S}$ is <u>not</u> normal

Issue: Wald tests with small n

The Wald test for a population mean μ relies on

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \approx N(0, 1)$$

- $Z_n \xrightarrow{d} N(0, 1)$ as $n \to \infty$
- But for small n, Z_n is not normal, even if $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

> Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. What is the exact distribution of $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s}$?

For large n, $t_{n-1} \approx N(0,1)$

t distribution

If $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

t distribution with $n-1$ df

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$$

<u>Definition</u>: Let $Z \sim N(0,1)$ and $V \sim \chi^2_d$ be independent.

Then $T = \dfrac{Z}{\sqrt{V/d}} \sim t_d$

<u>Apply</u>: $\dfrac{\sqrt{n}(\bar{X}-\mu)}{s} = \dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \cdot \dfrac{\sigma}{s}$

- $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$
- $\dfrac{s}{\sigma} = \sqrt{\dfrac{(n-1)s^2}{(n-1)\sigma^2}}$, and $\dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$
- $\bar{X} \perp\!\!\!\perp s^2$ (relies on $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$)

Intuition: $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{Y_i - \bar{Y}}{\sigma} \right)^2$

looks like $\sum_{i=1}^{n} Z_i^2$ $Z_i \sim N(0,1)$ (which is χ^2_n)

but we had to estimate μ $\Rightarrow$ lose a df for estimating μ w/ $\bar{X}$

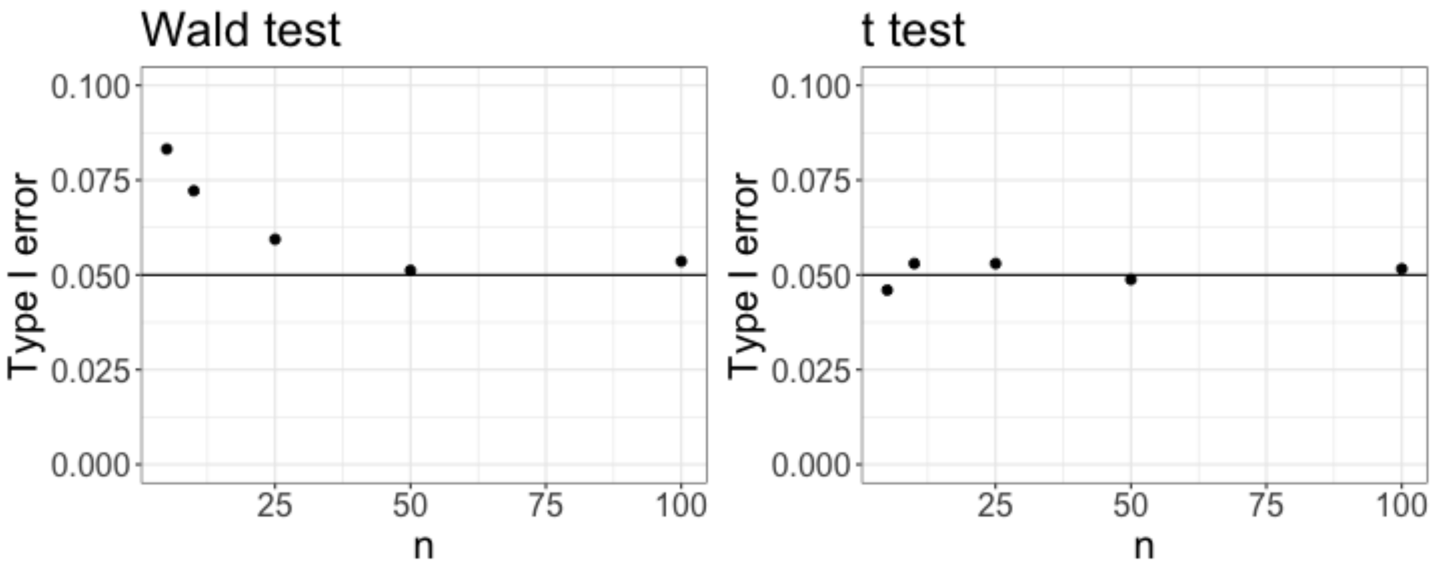
$\Rightarrow$ χ^2_{n-1}

Class activity, Part II

https://sta711-s23.github.io/class_activities/ca_lecture_21.html

$s =$ sd(x) $\qquad \bar{x} =$ mean(x)

Class activity



Example: two-sample t-test for a difference in means

Suppose that $X_1, \ldots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2)$ are independent samples. We want to test

$$\underset{\mu_1 - \mu_2 = 0}{H_0 : \mu_1 = \mu_2} \qquad \underset{\mu_1 - \mu_2 \neq 0}{H_A : \mu_1 \neq \mu_2}$$

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2,\ \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

$$\Rightarrow \frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1) \quad \text{under } H_0$$

$$s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{X})^2 \qquad s_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (y_i - \bar{Y})^2$$

$$\Rightarrow \mathbb{E}[s_1^2] = \mathbb{E}[s_2^2] = \sigma^2$$

Also, $\frac{(n_1 - 1) s_1^2}{\sigma^2} \sim \chi^2_{n_1 - 1}$ $\qquad \frac{(n_2 - 1) s_2^2}{\sigma^2} \sim \chi^2_{n_2 - 1}$

$$\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}$$

$$\frac{\bar{X}-\bar{Y}}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim N(0,1) \text{ under } H_0 \qquad \perp \qquad \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{\sigma^2}$$

$$\Rightarrow \quad \frac{\dfrac{\bar{X}-\bar{Y}}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}{\sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1+n_2-2)\sigma^2}}} \sim t_{n_1+n_2-2}$$

$$\Rightarrow \quad \frac{\bar{X}-\bar{Y}}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t_{n_1+n_2-2} \qquad S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1+n_2-2)}}$$

pooled estimate

Example: test for a population mean

Suppose $Y_1, \ldots, Y_n \overset{iid}{\sim} Bernoulli(p)$. We want to test

$$H_0 : p = p_0 \qquad H_A : p \neq p_0$$

Wald test:

> Why is a t-test not appropriate?

Example: logistic regression

$$Y_i \sim Bernoulli(p_i) \qquad \log\left(\frac{p_i}{1-p_i}\right) = \beta^T X_i$$

We want to test

$$H_0 : C\beta = \gamma_0 \qquad H_A : C\beta \neq \gamma_0$$

> Why is a t-test not appropriate?

Philosophical question

- If $X_1, \ldots, X_n$ are iid from a population with mean μ and variance σ^2, then $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \xrightarrow{d} N(0, 1)$
- If $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\frac{\sqrt{n}(\overline{X}_n - \mu)}{s} \sim t_{n-1}$
- **Position 1:** For any reasonable sample size, the test statistic is approximately normal. And we never really have data from a normal distribution, so the t distribution is an approximation anyway. So always use the normal distribution
- **Position 2:** We always have a finite sample size, so our test statistic is never truly normal. And the t distribution is more conservative than the normal (heavier tails). So always use the t distribution

> With which position do you agree?