Convergence of the MLE

Some more theorems about convergence

Continuous mapping theorem:

Let $X_1, X_2, \ldots$ be a sequence of random variables

Let g be a continuous function.

1. If $X_n \xrightarrow{d} X$ then $g(X_n) \xrightarrow{d} g(X)$
2. If $X_n \xrightarrow{P} X$ then $g(X_n) \xrightarrow{P} g(X)$ ← HW
3. If $X_n \xrightarrow{a.s.} X$ then $g(X_n) \xrightarrow{a.s.} g(X)$

Slutsky's theorem:

Let $\{X_n\}, \{Y_n\}$ be sequences of random variables, and suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} c$. Then

↑ constant

- $X_n + Y_n \xrightarrow{d} X + c$
- $X_n Y_n \xrightarrow{d} Xc$
- $\dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{c}$, provided c is invertible

Convergence of the MLE

Suppose that $Y_1, Y_2, Y_3, \ldots$ are iid with probability function $f(y \mid \theta)$, $\theta \in \mathbb{R}^d$

Let $\ell_n(\theta) = \sum_{i=1}^{n} \log f(Y_i \mid \theta)$, and $\hat{\theta}_n$ the MLE using first n observations. Let $I_1(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log f(Y_i \mid \theta)\right]$

(Fisher information for a single observation)

<u>Theorem</u>: Under regularity conditions,

(a) $\hat{\theta}_n \xrightarrow{P} \theta$ as $n \to \infty$ (consistency)

(b) $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, I_1^{-1}(\theta))$ as $n \to \infty$

(asymptotic normality)

We will prove (b) when $d = 1$

Proof sketch of (b): (when $d=1$)

① $\sqrt{n}(\hat{\theta}_n - \theta) \approx \dfrac{\frac{1}{\sqrt{n}} \ell_n'(\theta)}{-\frac{1}{n} \ell_n''(\theta)}$ (Taylor expansion)

② $\frac{1}{n} \ell_n''(\theta) \xrightarrow{P} -\mathcal{I}_1(\theta)$

③ $\frac{1}{\sqrt{n}} \ell_n'(\theta) \xrightarrow{d} N(0, \mathcal{I}_1(\theta))$

④ Apply Slutsky's: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \dfrac{1}{\mathcal{I}_1(\theta)} N(0, \mathcal{I}_1(\theta))$

$$= N(0, \mathcal{I}_1^{-1}(\theta))$$

$X \sim N(\mu, \sigma^2) \Rightarrow aX \sim N(a\mu, a^2\sigma^2)$

$a = \dfrac{1}{\mathcal{I}_1(\theta)}$ $\quad \sigma^2 = \mathcal{I}_1(\theta)$ $\quad a^2\sigma^2 = \dfrac{1}{\mathcal{I}_1(\theta)}$

Intermediate steps

Using results we have previously derived, argue that:

② $\frac{1}{n}\ell''(\theta|\mathbf{Y}) \xrightarrow{p} -\mathcal{I}_1(\theta)$ $\quad$ $\frac{1}{n}\ell_n''(\theta) \xrightarrow{p} -\mathcal{I}_1(\theta)$

③ $\frac{1}{\sqrt{n}}\ell'(\theta|\mathbf{Y}) \xrightarrow{d} N(0, \mathcal{I}_1(\theta))$ $\quad$ $\frac{1}{\sqrt{n}}\ell_n'(\theta) \xrightarrow{d} N(0, \mathcal{I}_1(\theta))$

Pf of ②:

$$\ell_n''(\theta) = \sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2} \log f(Y_i|\theta)$$

$$\underbrace{\frac{1}{n}\ell_n''(\theta)}_{\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)} \xrightarrow{p} \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(Y_i|\theta)\right] \quad \text{by WLLN}$$

$$= -\mathcal{I}_1(\theta) \quad //$$

Pf of 3 : $\qquad l_n'(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(Y_i|\theta)$

By CLT, $\qquad \sqrt{n}\left(\frac{1}{n} l_n'(\theta) - \mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i|\theta)\right]\right) \xrightarrow{d}$

$$N\left(0, \operatorname{Var}\left(\frac{\partial}{\partial\theta} \log f(Y_i|\theta)\right)\right)$$

$\mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i|\theta)\right] = 0$

$\operatorname{Var}\left(\frac{\partial}{\partial\theta} \log f(Y_i|\theta)\right) = \mathcal{I}_1(\theta)$

$\Rightarrow \qquad \sqrt{n}\left(\frac{1}{n} l_n'(\theta) - 0\right) \xrightarrow{d} N(0, \mathcal{I}_1(\theta))$

$\Rightarrow \qquad \frac{1}{\sqrt{n}} l_n'(\theta) \xrightarrow{d} N(0, \mathcal{I}_1(\theta)) \qquad //$

Full proof:

① If $\hat{\theta}_n$ is MLE, then $l_n'(\hat{\theta}_n) = 0$

If $\hat{\theta}_n \approx \theta$ (which holds because $\hat{\theta}_n \xrightarrow{p} \theta$), then

$$0 = l_n'(\hat{\theta}_n) \approx l_n'(\theta) + (\hat{\theta}_n - \theta)\, l_n''(\theta)$$

$$\Rightarrow \quad \hat{\theta}_n - \theta \approx \frac{l_n'(\theta)}{-l_n''(\theta)}$$

$$\Rightarrow \sqrt{n}(\hat{\theta}_n - \theta) \approx \frac{\sqrt{n}\,(l_n'(\theta))}{-l_n''(\theta)}$$

$$= \frac{\frac{1}{\sqrt{n}}\, l_n'(\theta)}{-\frac{1}{n}\, l_n''(\theta)} \xrightarrow{d} N(0, I_1^{-1}(\theta))$$

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Regularity conditions