Convergence of random variables

Warm-up

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} Uniform(0,1)$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

Working with a neighbor, argue that $X_{(n)} \xrightarrow{p} 1$.

Pf: Let $\varepsilon > 0$. WTS $P(|X_{(n)} - 1| \geq \varepsilon) \to 0$ as $n \to \infty$

$$P(|X_{(n)} - 1| \geq \varepsilon) = 1 - P(-\varepsilon \leq X_{(n)} - 1 \leq \varepsilon)$$

$$= 1 - P(1 - \varepsilon \leq X_{(n)} \leq 1 + \varepsilon)$$

Can $X_{(n)} > 1+\varepsilon$?

No! $X_{(n)} \leq 1+\varepsilon$

$$= 1 - P(X_{(n)} \geq 1 - \varepsilon)$$

$$= P(X_{(n)} \leq 1 - \varepsilon) = P(X_1 \leq 1-\varepsilon, X_2 \leq 1-\varepsilon, \ldots, X_n \leq 1-\varepsilon)$$

$$= (1-\varepsilon)^n \to 0 \text{ as } n \to \infty \quad //$$

$f_Y(y|\theta) = \theta e^{-\theta y}$ $F_Y(t) = 1 - e^{-\theta t}$

Warm-up

$\theta = 1$

$\Rightarrow F_Y(t) = \boxed{1 - e^{-t}}$

Suppose that $X_1, X_2, \ldots \overset{iid}{\sim} Uniform(0,1)$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

> Show that $n(1 - X_{(n)}) \xrightarrow{d} Y$, where $Y \sim Exp(1)$.

Pf: WTS $F_{n(1-X_{(n)})}(t) \to F_Y(t)$ $\forall\, t$ where $F_Y(t)$ is continuous

$$F_{n(1-X_{(n)})}(t) = P(n(1-X_{(n)}) \le t)$$

$$= P(1 - X_{(n)} \le \tfrac{t}{n}) = P(X_{(n)} \ge 1 - \tfrac{t}{n})$$

$$= 1 - P(X_{(n)} \le 1 - \tfrac{t}{n})$$

$$= 1 - (1 - \tfrac{t}{n})^n$$

$$\to 1 - e^{-t} \quad \text{as } n \to \infty$$

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$$\boxed{\lim_{n \to \infty} (1 + \tfrac{x}{n})^n = e^x}$$

$x = -t$

Relationships between types of convergence

(a) If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{P} X$

(b) If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{d} X$

(c) If $X_n \xrightarrow{d} c$, where c is a constant, then $X_n \xrightarrow{P} c$

Ex: ($X_n \xrightarrow{P} X$ does not imply $X_n \xrightarrow{a.s.} X$)

Let $U \sim \text{Uniform}(0,1)$, and construct the following sequence:

$X_1 = \mathbb{1}\{U \in [0,1]\}$, $X_2 = \mathbb{1}\{U \in [0, \frac{1}{2}]\}$, $X_3 = \mathbb{1}\{U \in [\frac{1}{2}, 1]\}$,

$X_4 = \mathbb{1}\{U \in [0, \frac{1}{4}]\}$, $X_5 = \mathbb{1}\{U \in [\frac{1}{4}, \frac{1}{2}]\}$, ...

$X_n = \mathbb{1}\{U \in [\frac{j}{2^k}, \frac{j+1}{2^k}]\}$ $\quad k = \lfloor \log_2(n) \rfloor$, $j = 2^k - n$

(1) $X_n \xrightarrow{P} 0$ $\quad$ Pf: $P(|X_n| > \varepsilon) = P(U \in [\frac{j}{2^k}, \frac{j+1}{2^k}]) = \frac{1}{2^k} \to 0$ as $n \to \infty$

(2) $X_n \overset{a.s.}{\not\to} 0$ $\quad$ Pf: $\lim_{n\to\infty} |X_n - 0|$ does not exist. For any u, there are infinitely many n st $X_n = 1$

<u>Want</u>: if $X_n \xrightarrow{a.s.} 0$, then $\lim_{n\to\infty} |X_n - 0| = 0$

i.e., $\forall\ \delta > 0$, $\exists\ n_0$ st $\forall\ n \geq n_0$, $|X_n - 0| \leq \delta$

<u>But</u>: let $\delta = \frac{1}{2}$. $\forall\ n_0$, $\exists\ n > n_0$ st $X_n = 1$

$\Rightarrow \exists\ n > n_0$ st $|X_n - 0| > \delta$

$\Rightarrow \lim_{n\to\infty} |X_n - 0|$ does not exist

Proof of (b) $(X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X)$

Proof sketch: WTS $F_{X_n}(x) \to F_X(x)$ $\forall x$ where F is continuous, as $n \to \infty$

we know $F(x-\varepsilon) \le F(x) \le F(x+\varepsilon)$ $\forall \varepsilon > 0$ (cdf is non-decreasing)

Also, if x is a continuity point of F, then $\lim_{\varepsilon \to 0} F(x-\varepsilon) = \lim_{\varepsilon \to 0} F(x+\varepsilon) = F(x)$

So, it suffices to show that $\forall \varepsilon > 0$, $\boxed{F(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) \le F(x+\varepsilon)}$

Formal proof: Let $\varepsilon > 0$, and x be a continuity point of F.

$$F_{X_n}(x) = P(X_n \le x) = \underbrace{P(X_n \le x, X \le x+\varepsilon)}_{\le P(X \le x+\varepsilon)} + \underbrace{P(X_n \le x, X > x+\varepsilon)}_{\le P(|X_n - X| > \varepsilon)}$$

(disjoint partition)

If $X_n \le x$ and $X > x+\varepsilon$
$\Rightarrow X > X_n$
$|X - X_n| > (x+\varepsilon) - x$
$= \varepsilon$

$$\Rightarrow F_{X_n}(x) \le F(x+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0 \text{ as } n\to\infty}$$

b/c $X_n \xrightarrow{P} X$

Similarly, $F(x-\varepsilon) - \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0 \text{ as } n \to \infty} \le F_{X_n}(x)$

$$\Rightarrow F(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) \le F(x+\varepsilon)$$

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Proof of (c) $\quad$ ($X_n \xrightarrow{d} c$ then $X_n \xrightarrow{p} c$)

WTS $\quad P(|X_n - c| > \varepsilon) \to 0 \quad \forall \varepsilon > 0$

Let $\varepsilon > 0$. $\quad P(|X_n - c| > \varepsilon) = P(X_n < c - \varepsilon \text{ or } X_n > c + \varepsilon)$

$$= P(X_n < c-\varepsilon) + P(X_n > c+\varepsilon)$$

$$\leq P(X_n \leq c-\varepsilon) + P(X_n > c+\varepsilon)$$

$$= F_{X_n}(c-\varepsilon) + (1 - F_{X_n}(c+\varepsilon))$$

$X_n \xrightarrow{d} c$: $\quad F_{X_n}(c-\varepsilon) \to F(c-\varepsilon) = 0 \quad$ (point mass at c)

$\qquad F_{X_n}(c+\varepsilon) \to F(c+\varepsilon) = 1$

$$\Rightarrow \lim_{n\to\infty} P(|X_n - c| > \varepsilon) \leq$$

$$F(c-\varepsilon) + (1 - F(c+\varepsilon))$$

$$= 0 + (1-1)$$

$$= 0$$

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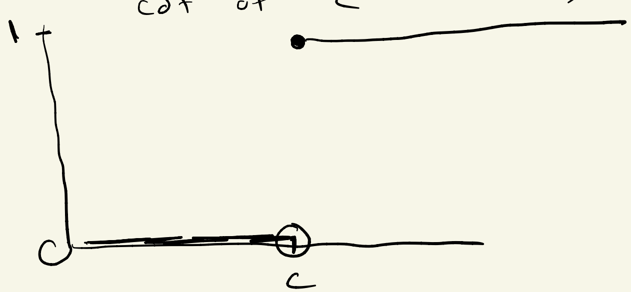
cdf of c (point mass)



Continuous mapping theorem

Slutsky's theorem