Convergence of random variables

Where we're heading

$$\hat{\beta} \underset{\sim}{\approx} N(\beta, \underset{\sim}{I}^{-1}(\beta))$$

- Need $\hat{\beta} \to \beta$ as $n \to \infty$
- Need $\hat{\beta} \underset{\sim}{\approx}$ Normal
- Need $\mathrm{Var}(\hat{\beta}) \underset{\sim}{\approx} \underset{\sim}{I}^{-1}(\beta)$

Convergence in probability

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in probability* to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

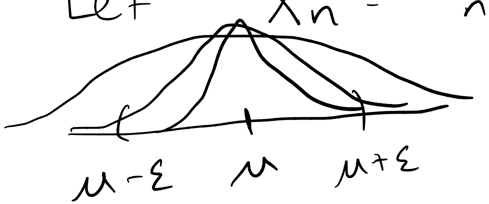
We write $X_n \xrightarrow{p} X$.

Example: (Weak law of large numbers)

Let $X_1, X_2, \ldots$ be iid with $\mathbb{E}[X_i] = \mu$ and

$\text{Var}(X_i) = \sigma^2 < \infty$.

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then $\bar{X}_n \xrightarrow{p} \mu$



$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{(iid)}$$

WLLN

$$Var(\bar{X}_n) = \frac{1}{n^2} \cdot \sum_i Var(X_i) = \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

Theorem: Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then

$$\bar{X}_n \xrightarrow{p} \mu$$

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum_i X_i\right] = \frac{1}{n} \cdot \sum_i \mathbb{E}[X_i] = \mu$$

> Working with your neighbor, apply Chebyshev's inequality to prove the WLLN.

Pf: Let $\varepsilon > 0$. WTS $P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$.

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} \quad \text{(Chebyshev)}$$

$$Var(\bar{X}_n) = \frac{\sigma^2}{n} \Rightarrow 0 \leq P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0$$

(as $n \to \infty$)

Another example

Let $U \sim Uniform(0,1)$, and let $X_n = \sqrt{n}\,\mathbb{I}\{U \leq 1/n\}$.

> Show that $X_n \xrightarrow{p} 0$.

Pf: Let $\varepsilon > 0$. WTS $P(|X_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$

$P(|X_n - 0| \geq \varepsilon) = P(\sqrt{n}\,\mathbb{1}\{U \leq \frac{1}{n}\} \geq \varepsilon)$

For sufficiently large n, $\sqrt{n} \geq \varepsilon$

$\Rightarrow \sqrt{n}\,\mathbb{1}\{U \leq \frac{1}{n}\} \geq \varepsilon$

if $\mathbb{1}\{U \leq \frac{1}{n}\} = 1$

$\Rightarrow$ if $U \leq \frac{1}{n}$

$= P(U \leq \frac{1}{n})$

$= \dfrac{\frac{1}{n} - 0}{1 - 0} = \frac{1}{n} \to 0$

(as $n \to \infty$)

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Almost sure convergence

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges almost surely* to a random variable X if, for every $\varepsilon > 0$,

$$P(\lim_{n\to\infty} |X_n - X| < \varepsilon) = 1$$

We write $X_n \stackrel{a.s.}{\to} X$.

Example: (Strong law of large numbers)

Let $X_1, X_2, \ldots$ be iid with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then $\bar{X}_n \xrightarrow{a.s.} \mu$

(proof sketch in C & B)

Convergence in distribution

Definition: A sequence of random variables $X_1, X_2, \ldots$ *converges in distribution* to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points where $F_X(x)$ is continuous. We write $X_n \xrightarrow{d} X$.

Example: (Central limit theorem)

Let $X_1, X_2, \ldots$ be a sequence of iid random variables whose mgfs exist in a neighborhood of 0. Let $\mathbb{E}[X_i] = \mu$, and $Var(X_i) = \sigma^2$. Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \qquad Z \sim N(0,1)$$

Multivariate CLT: $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0, \Sigma)$
$\Sigma = Var(X_i)$

Another example

Let $X \sim N(0,1)$, and let $X_n = -X$ for $n = 1, 2, 3, \ldots$.

> Show that $X_n \xrightarrow{d} X$, but X_n does *not* converge to X in probability.

Pf: WTS $F_{X_n}(x) \to F_X(x)$ $\forall$ x where F is continuous

$X_n \sim N(0,1)$

$F_{X_n}(x) \equiv F_X(x) \quad \forall x \quad \Rightarrow \quad X_n \xrightarrow{d} X$

But: $P(|X_n - X| \geq \varepsilon) = P(|2X| \geq \varepsilon)$

$= P(|X| \geq \frac{\varepsilon}{2}) \not\to 0$

$\Rightarrow X_n \overset{P}{\not\to} X$

Relationships between types of convergence