Inequalities and Asymptotics

Wald tests for single parameters

Logistic regression model for the dengue data:

$$Y_i \sim Bernoulli(p_i)$$

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 WBC_i + \beta_2 PLT_i$$

> Researchers want to know if there is a relationship between white blood cell count and the probability a patient has dengue, after accounting for platelet count. What hypotheses should the researchers test?

$H_0: \beta_1 = 0$

$H_A: \beta_1 \neq 0$

Need $\hat{\beta} \approx$ Normal

Wald tests for single parameters

Wald test: using approximate normal distribution

```
m1 <- glm(Dengue ~ WBC + PLT, data = dengue,
          family = binomial)
summary(m1)
```

```
...
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  2.6415063  0.1213233   21.77   <2e-16 ***
## WBC         -0.2892904  0.0134349  -21.53   <2e-16 ***
## PLT         -0.0065615  0.0005932  -11.06   <2e-16 ***
## ---
...
```

value hypothesized under H_0

Test statistic: $Z = \dfrac{\hat{\beta}_1 - \beta_1^0}{SE(\hat{\beta}_1)}$

$= \dfrac{-0.289 - 0}{0.013} \approx -21.53$

p-value: $P(|Z| > |-21.53|)$ $\quad Z \sim N(0,1)$

What this requires

$$\hat{\beta} \approx N(\beta, \mathcal{I}^{-1}(\beta))$$

- Need $\hat{\beta} \to \beta$ as $n \to \infty$
- Need $\hat{\beta} \approx N$
- Need $\text{Var}(\hat{\beta}) \approx \mathcal{I}^{-1}(\beta)$

What we need to do

1. Preliminary machinery:
   - probability inequalities
   - types of convergence for random variables
   - theorems about convergence
2. Properties of maximum likelihood estimators
   - consistency ($\hat{\theta} \to \theta$)
   - asymptotic normality ($\hat{\theta} \approx$ normal)
3. Wald tests & confidence intervals!

Markov's inequality

Theorem: Let Y be a non-negative random variable, and suppose that $\mathbb{E}[Y]$ exists. Then for any $t > 0$,

$$P(Y \geq t) \leq \frac{\mathbb{E}[Y]}{t}$$

<u>Pf</u>: $\mathbb{E}(Y) = \int_0^\infty y f(y)\, dy$

$$= \int_0^t y f(y)\, dy + \int_t^\infty y f(y)\, dy$$

$$\geq \int_t^\infty y f(y)\, dy \geq t \int_t^\infty f(y)\, dy = t P(Y > t)$$

//

$|Y-\mu| \geq t \iff (Y-\mu)^2 \geq t^2$

Chebyshev's inequality

Theorem: Let Y be a random variable, and let $\mu = \mathbb{E}[Y]$ and $\sigma^2 = Var(Y)$. Then

$\mathbb{E}[(Y-\mu)^2] = \sigma^2$

$$P(|Y - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

> With your neighbor, apply Markov's inequality to prove Chebyshev's inequality.

(markov's)

Pf: $P(|Y-\mu| \geq t) = P((Y-\mu)^2 \geq t^2) \leq \frac{\mathbb{E}[(Y-\mu)^2]}{t^2} = \frac{\sigma^2}{t^2}$ //

Cauchy-Schwarz inequality

Theorem: For any two random variables X and Y,

(Casella & Berger, Theorem 4.7.3)

$$|\mathbb{E}[XY]| \leq \mathbb{E}|XY| \leq (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2}$$

Example: The *correlation* between X and Y is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\underbrace{\sqrt{Var(X)}}_{\mathbb{E}[(X-\mu_X)^2]}\underbrace{\sqrt{Var(Y)}}_{\mathbb{E}[(Y-\mu_Y)^2]}}$$

Working with a neighbor, use the Cauchy-Schwarz inequality to prove that $-1 \leq \rho(X,Y) \leq 1$.

(Cauchy Schwarz)

$$|Cov(X,Y)| = |\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]| \leq \mathbb{E}[(X-\mu_X)^2]^{\frac{1}{2}}\,\mathbb{E}[(Y-\mu_Y)^2]^{\frac{1}{2}}$$

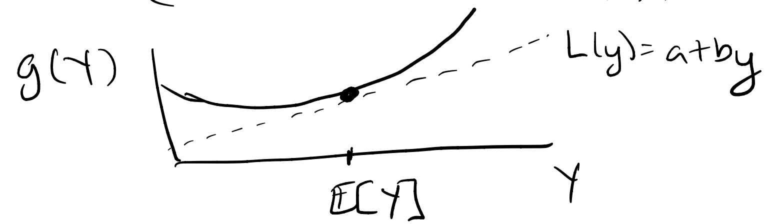
$$= \sqrt{Var(X)}\sqrt{Var(Y)}$$

$$\Rightarrow \quad |\rho(X,Y)| \leq 1 \quad \Rightarrow \quad -1 \leq \rho(X,Y) \leq 1$$

//

Ex: $g(x) = x^2$ (convex) $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2 \Rightarrow Var(X) \geq 0$

Jensen's inequality



Theorem: For any random variable Y, if g is a convex function, then

$$\mathbb{E}[g(Y)] \geq g(\mathbb{E}[Y])$$

Recall: g is convex if $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$
$\forall x, y$ and $\forall \lambda \in (0,1)$

Pf: Let $L(y)$ be the tangent line to $g(y)$ at the point $y = \mathbb{E}[Y]$
$L(y) = a + by$ for some a, b

By convexity, $g(y) \geq a + by \quad \forall y$

$\Rightarrow \mathbb{E}[g(Y)] \geq \mathbb{E}[L(Y)] = a + b\,\mathbb{E}[Y]$

$= L(\mathbb{E}[Y]) = g(\mathbb{E}[Y])$

$\Rightarrow \mathbb{E}[g(Y)] \geq g(\mathbb{E}[Y])$ //